MATH 119: Final

Name: Key

Directions:

* Show your thought process (commonly said as ”show your work”) when solving each problem for full credit.
* If you do not know how to solve a problem, try your best and/or explain in English what you would do.
* Good luck!

Problem	Score	Points
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
		120

1. Short answer questions:

   (a) Suppose you try distributing

$$(x+y)^2z^2 = (xz^2 + yz^2)^2$$

   Why is this incorrect?

   Because $(x+y)^2 z^2 = (x+y)(x+y)z^2$; you distributed the z^2 into both factors of $(x+y)$. The distributive law says you can only distribute into one factor of $(x+y)$.

   (b) Suppose you cancel out the x's to simplify

$$\frac{3+x}{x} = \frac{3+1}{1} = 4$$

   Why is this incorrect?

   Because x is a term in the context of the numerator. Fraction law #5 says $\frac{a \cdot \cancel{c}}{b \cdot \cancel{c}} = \frac{a}{b}$; you can only cancel if c is a factor in the context of the entire numerator and denominator.

   (c) You try simplifying by distributing:

$$[(x-1)^2 + (x+1)]^3 = (x-1)^5 + (x+1)^3 = x^5 - 1^5 + x^3 + 1^3 = x^5 + x^3$$

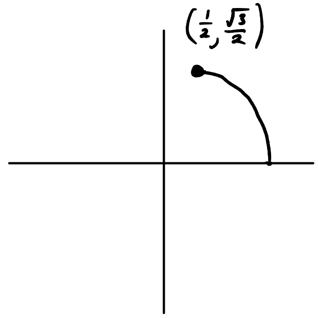
   Circle the two types of mistakes you made and explain why they are mistakes.

   In purple:

   (1) Distributed the power of 3 to the term $(x-1)^2$

   (2) If you had $((x-1)^2)^3$, you need to multiply 2 and 3 not add them.

   In red: Distributed exponent to terms. Can only distribute to factors!

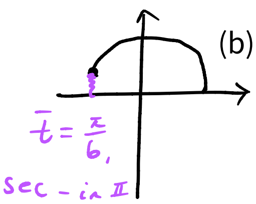


2. Evaluate the following:

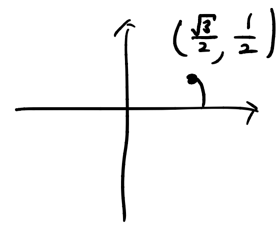
(a) $\sin\left(\frac{\pi}{3}\right) = \boxed{\frac{\sqrt{3}}{2}}$

convert to radians

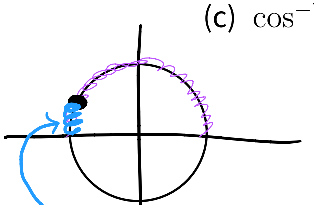
(b) $\sec(150°) = \sec\left(\cancel{150°}^{5} \cdot \frac{\pi}{\cancel{180°}_{6}}\right) = \sec\left(\frac{5\pi}{6}\right)$



$= -\sec\left(\frac{\pi}{6}\right) = -\frac{1}{\frac{\sqrt{3}}{2}} = -\frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \boxed{-\frac{2\sqrt{3}}{3}}$



(c) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$



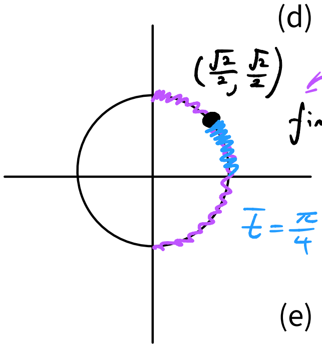
Look for x-coordinate of $-\frac{\sqrt{3}}{2}$ in the shaded purple portion.

$\overline{t} = \frac{\pi}{6}$ in order to get $\frac{\sqrt{3}}{2}$ cos − in II

$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \boxed{\frac{5\pi}{6}}$

to walk to that point.

(d) $\tan\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$



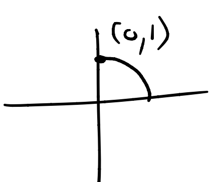
find t value for y-coordinate in the shaded portion

$\tan\left(\sin^{-1}\frac{\sqrt{2}}{2}\right) = \tan\left(\frac{\pi}{4}\right)$

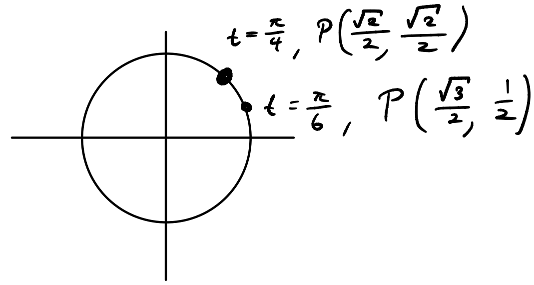
$= \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}$

$= \boxed{1}$

(e) $\sin\left(\frac{5\pi}{2}\right)$



$= \sin\left(\underbrace{2\pi}_{\text{walk one circle}} + \underbrace{\frac{\pi}{2}}_{\text{then a quarter}}\right) = \boxed{1}$



(f) $\sin 75° = \sin(30° + 45°)$

$30° \cdot \frac{\pi}{180°} = \frac{\pi}{6}$

$= \sin\left(\frac{\pi}{6} + \frac{\pi}{4}\right)$

$45° \cdot \frac{\pi}{180°} = \frac{\pi}{4}$

addition formula

$= \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right) = \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}$

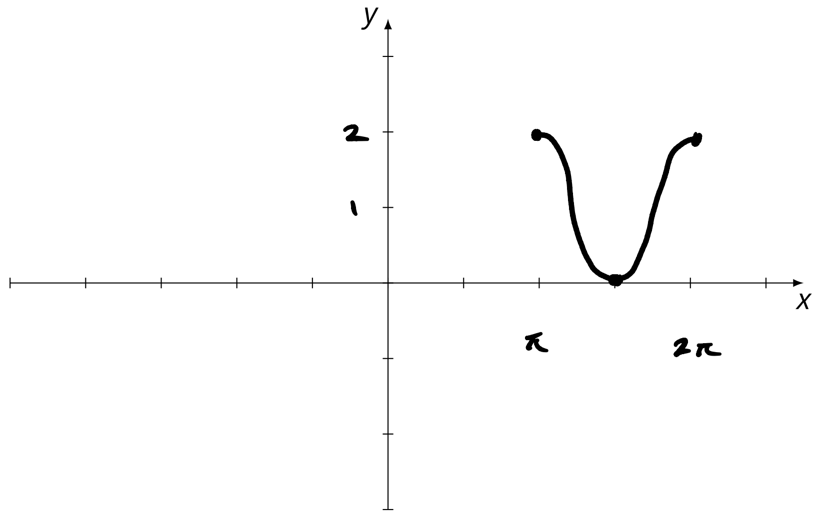
$= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \boxed{\frac{\sqrt{2}+\sqrt{6}}{4}}$

3. Given the function

not coefficient of 1

$$f(x) = 1 + \cos(2x - 2\pi) = 1 + \cos(2(x - \pi))$$

(a) Graph one period of $f(x)$ using transformations.



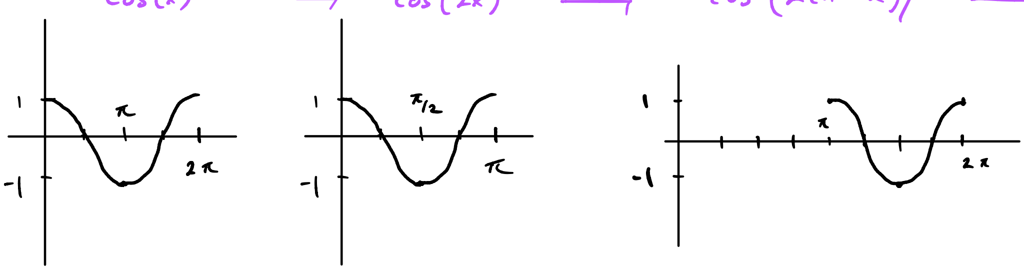
Base function $g(x) = \cos(x)$

$h(x) = g(2x) = \cos(2x)$ horizontal shrink $\frac{1}{2}$

$k(x) = h(x - \pi) = \cos(2(x - \pi))$ horizontal shift π units to the right

$f(x) = 1 + k(x) = 1 + \cos(2(x - \pi))$ vertical shift up 1 unit.

$\cos(x) \longrightarrow \cos(2x) \longrightarrow \cos(2(x - \pi)) \longrightarrow 1 + \cos(2(x - \pi))$

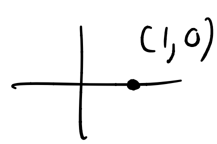


see above

(b) What is $f(\pi)$?

$$f(\pi) = 1 + \cos(2\pi - 2\pi)$$

$$= 1 + \cos(0)$$



$$= 1 + 1$$

$$= \boxed{2}$$

4. Suppose $f(x) = x - x^2$

(a) A person tries to find $f(x+h)$ by writing

$$f(x+h) = x - x^2 + h$$

This is wrong. What expression (involving $f(x)$) did the person actually write down?

$\underbrace{x - x^2}_{f(x)} \underbrace{+ h}_{+h}$ so $\boxed{f(x) + h}$

(b) The person then tries again:

$$f(x+h) = x + h - x + h^2$$

Explain the reason why this is also incorrect.

$x+h$ is two terms. In $f(x) = x - x^2$, replacing the x in $-x^2$ with $x+h$ requires parenthesis since two terms are being subtracted and taken to a power.

(c) Your turn: Evaluate $f(x+h)$ and fully simplify.

$f(x+h) = (x+h) - (x+h)^2$

this is often forgotten as well. You are subtracting the entirety of three terms.

$\overset{\text{special product}}{=} x + h - (x^2 + 2xh + h^2)$

$= \boxed{x + h - x^2 - 2xh - h^2}$

(d) In general, when you are substituting two or more terms into **(a)** a variable with a power or **(b)** that variable being subtracted, what do you need to not forget?

Do not forget to put parenthesis around the entire expression you are subtracting or taking the power of.

5. Solve the equation for θ. Check your work if necessary.

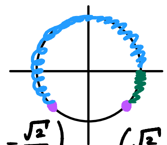
(a) $\sqrt{2}\sin\theta + 1 = 0$

isolate $\sin\theta$.

$$\sqrt{2}\sin\theta = -1$$

$$\sin\theta = -\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

① in one period.



$\pi + \frac{\pi}{4} = \frac{5\pi}{4}$ $\quad -\frac{\pi}{4}$ or $\frac{7\pi}{4}$

$$\theta = \frac{5\pi}{4}, \quad \theta = \frac{7\pi}{4}$$

② account for periodicity

period of $\sin\theta$ is 2π long.

$$\boxed{\theta = \frac{5\pi}{4} + 2k\pi, \quad \theta = \frac{7\pi}{4} + 2k\pi, \quad k \in \mathbb{Z}}$$

(b) $\sin^2\theta = 4 - 2\cos^2\theta$

Convert to one trig function

$\cos^2\theta = 1 - \sin^2\theta$

pythagorean identity

$$\sin^2\theta = 4 - 2(1 - \sin^2\theta)$$

$$\sin^2\theta = 4 - 2 + 2\sin^2\theta \qquad \text{distributive law}$$

$$0 = 2 - \sin^2\theta \qquad \text{collect like term}$$

$$\sqrt{\sin^2\theta} = \pm\sqrt{2} \qquad \text{isolate } \sin^2\theta$$

$$\sin\theta = \pm\sqrt{2}$$

$$\sin\theta = \sqrt{2}, \quad \sin\theta = -\sqrt{2}$$

Since $\sqrt{2} \approx 1.414$ and the range of sine is $[-1, 1]$

no θ exists for both equations.

$\boxed{\text{No solution}}$

6. Prove these identities algebraically:

(a) $\dfrac{\sin\theta}{\tan\theta} = \cos\theta$

$\dfrac{1}{\frac{b}{c}} = 1 \cdot \dfrac{c}{b}$

$$LHS = \frac{\sin\theta}{\tan\theta} = \frac{\sin\theta}{\frac{\sin\theta}{\cos\theta}} = \cancel{\sin\theta} \cdot \frac{\cos\theta}{\cancel{\sin\theta}} \overset{\text{frac}}{\underset{\text{law \#5}}{=}} \cos\theta = RHS$$

(b) $\dfrac{\cos x}{\sec x} + \dfrac{\sin x}{\csc x} = 1$

$\dfrac{1}{\frac{b}{c}} = 1 \cdot \dfrac{c}{b}$

$$LHS = \frac{\cos x}{\sec x} + \frac{\sin x}{\csc x} \overset{\text{trig}}{\underset{\text{function definition}}{=}} \frac{\cos x}{\frac{1}{\cos x}} + \frac{\sin x}{\frac{1}{\sin x}} = \cos x \cdot \cos x + \sin x \cdot \sin x$$

$$= \cos^2 x + \sin^2 x$$

$$\overset{\text{pythagorean}}{\underset{\text{identity}}{=}} 1$$

$$= RHS$$

(c) $\cos^4 x - \sin^4 x = \cos 2x$

$$LHS = \cos^4 x - \sin^4 x = \underbrace{(\cos^2 x - \sin^2 x)}_{\cos(2x)}\underbrace{(\cos^2 x + \sin^2 x)}_{1}$$

$$= \cos(2x) \cdot 1$$

$$= \cos 2x$$

$$= RHS$$

Let $A = \cos^2 x$, $B = \sin^2 x$

Then $\cos^4 x - \sin^4 x \overset{\text{LoE}}{=} (\cos^2 x)^2 - (\sin^2 x)^2$

$= A^2 - B^2$

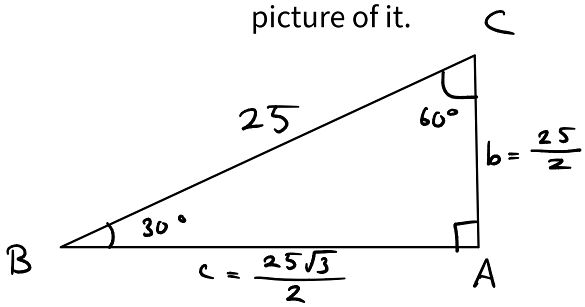
Special product $= (A-B)(A+B)$

$= \boxed{(\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x)}$

you could also use 30°-60°-90° triangle and notice this triangle is similar.

7. Answer the following:

(a) A triangle ABC has $\angle A = 90°$, $\angle B = 30°$ and $A = 25$. Solve the triangle and draw a picture of it.



For $\angle C$:

$\angle A + \angle B + \angle C = 180°$

$90° + 30° + \angle C = 180°$

$\boxed{\angle C = 60°}$

For c: $\cos(30°) = \frac{c}{25}$

$c = 25\cos(30°)$

$30° \cdot \frac{\pi}{180°} = \frac{\pi}{6}$

$= 25\cos\left(\frac{\pi}{6}\right) = 25 \cdot \frac{\sqrt{3}}{2} = \boxed{\frac{25\sqrt{3}}{2}}$

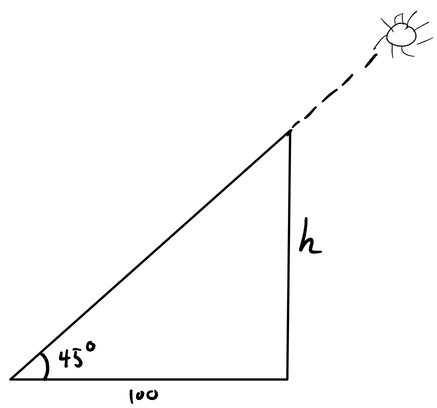
For b: $\sin(30°) = \frac{b}{25}$ ($\frac{\pi}{6}$)

$b = 25\sin\left(\frac{\pi}{6}\right)$

$= 25 \cdot \frac{1}{2}$

$= \frac{25}{2}$

(b) A sequoia tree casts a shadow 100 feet long. Find the height of the tree if the angle of elevation of the sun is 45°.



$45° \cdot \frac{\pi}{180°} = \frac{\pi}{4}$

$\tan(45°) = \frac{h}{100}$

$h = 100\tan\left(\frac{\pi}{4}\right)$

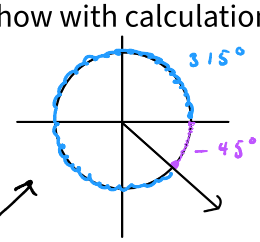
$= 100 \cdot \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}$

$= 100 \cdot 1$

$= \boxed{100 \text{ feet}}$

(c) Are $\frac{-\pi}{4}$ rad and 315° coterminal? Show with calculations.

$-\frac{\cancel{\pi}}{\cancel{4}} \cdot \frac{\cancel{180°}^{\,45°}}{\cancel{\pi}} = -45°$



A full circle is highlighted since $315° + 45° = 360°$

The terminal sides therefore lie on top of each other so they are coterminal.

8. Answer the following:

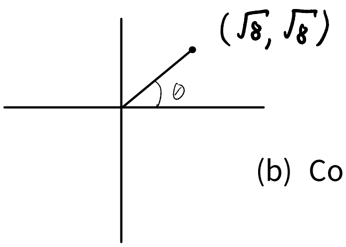
(a) Convert $(\sqrt{8}, \sqrt{8})$ into polar coordinates.

$\uparrow x \quad \uparrow y$

$$r = \sqrt{x^2 + y^2} = \sqrt{(\sqrt{8})^2 + (\sqrt{8})^2} = \sqrt{8+8} = \sqrt{16} = 4$$

$$\tan\theta = \frac{\sqrt{8}}{\sqrt{8}} = 1$$

$$\boxed{\left(4, \frac{\pi}{4}\right)}$$



looking here $\theta = \tan^{-1}(1) = \frac{\pi}{4}$, only way to point in quadrant 1.

(b) Convert $r = \dfrac{1}{1 + \sin\theta}$ into rectangular form.

here is a r you can use

missing r to use $r\sin\theta$

$$(1+\sin\theta) \cdot r = \frac{1}{\cancel{1+\sin\theta}} \cdot \cancel{(1+\sin\theta)}$$

$$r + r\sin\theta = 1 \quad \text{dist law}$$

$$r + y = 1$$

$$r = 1 - y \quad \text{isolate } r$$

$$r^2 = (1-y)^2 \quad \text{create } r^2$$

$$\boxed{x^2 + y^2 = (1-y)^2} \quad \text{in order to use } r^2 = x^2 + y^2$$

(c) Convert $r = 6\cos\theta$ into rectangular form.

both missing r to use formulas

$$r \cdot r = 6\cos(\theta) \cdot r$$

$$r^2 = 6r\cos(\theta)$$

$$\boxed{x^2 + y^2 = 6x}$$

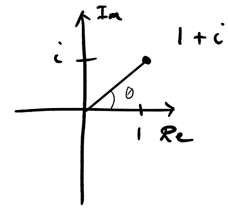
9. Answer the following:

(a) Write $1+i$ in polar form.

$a+bi$, $a=1$, $b=1$

$r=\sqrt{a^2+b^2}=\sqrt{1^2+1^2}=\boxed{\sqrt{2}}$

$\tan\theta=\frac{b}{a}=\frac{1}{1}=1$ ←



$\tan\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}/2}{\sqrt{2}/2}=1$

and $\frac{\pi}{4}$ points in the right direction.

$\therefore$, $\boxed{\theta=\frac{\pi}{4}.}$

So $\boxed{\begin{aligned}1+i&=r(\cos\theta+i\sin\theta)\\&=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\end{aligned}}$

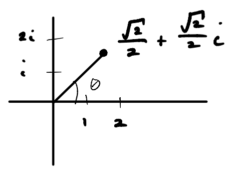
(b) Evaluate $\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)^{12}$

$a+bi$

$a=\frac{\sqrt{2}}{2}$, $b=\frac{\sqrt{2}}{2}$

$r=\sqrt{\left(\frac{\sqrt{2}}{2}\right)^2+\left(\frac{\sqrt{2}}{2}\right)^2}\overset{\text{LoE}}{=}\sqrt{\frac{(\sqrt{2})^2}{2^2}+\frac{(\sqrt{2})^2}{2^2}}=\sqrt{\frac{2}{4}+\frac{2}{4}}=\sqrt{\frac{4}{4}}=\sqrt{1}=\boxed{1}$

$\tan\theta=\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}=1$ ←



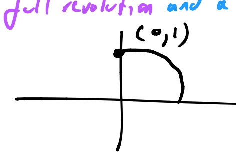
Same calculation for θ as previous problem. $\boxed{\theta=\frac{\pi}{4}}$

So $\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)^{10}\overset{\text{DeMoivre}}{=}1^{10}\cdot\left(\cos\left(\overset{5}{\cancel{10}}\cdot\frac{\pi}{\underset{2}{\cancel{4}}}\right)+i\sin\left(\overset{5}{\cancel{10}}\cdot\frac{\pi}{\underset{2}{\cancel{4}}}\right)\right)$

$=1\cdot\left(\cos\left(\frac{5\pi}{2}\right)+i\sin\left(\frac{5\pi}{2}\right)\right)$ ← $\frac{5\pi}{2}=\underbrace{2\pi}+\underbrace{\frac{\pi}{2}}$

one full revolution and a quarter

$=\cos\left(\frac{5\pi}{2}\right)+i\sin\left(\frac{5\pi}{2}\right)$



$=0+i\cdot 1$

$=\boxed{i}$

10. Simplify the following trigonometric expressions:

(a) $\sin 20^\circ \cos 40^\circ + \cos 20^\circ \sin 40^\circ$

addition formula

$$\overset{\text{addition formula}}{=} \sin(60^\circ)$$

$$= \sin\left(\frac{\pi}{3}\right)$$

$$= \boxed{\frac{\sqrt{3}}{2}}$$

generates 2 terms.

(b) $\dfrac{\sin(x+y) - \overbrace{\sin(x-y)}^{\text{generates 2 terms.}}}{\cos(x+y) + \cos(x-y)}$

$$= \frac{\sin(x)\cos(y) + \cos(x)\sin(y) - \left(\sin(x)\cos(y) - \cos(x)\sin(y)\right)}{\cos(x)\cos(y) - \cancel{\sin(x)\sin(y)} + \cos(x)\cos(y) + \cancel{\sin(x)\sin(y)}}$$

four applications of addition/subtraction formula

$$\overset{\text{dist}}{=} \frac{\cancel{\sin(x)\cos(y)} + \cos(x)\sin(y) - \cancel{\sin(x)\cos(y)} + \cos(x)\sin(y)}{2\cos(x)\cos(y)}$$

$$= \frac{\cancel{2}\cdot\cancel{\cos(x)}\cdot\sin(y)}{\cancel{2}\cdot\cancel{\cos(x)}\cdot\cos(y)}$$

$$\overset{\text{frac law 5}}{=} \frac{\sin(y)}{\cos(y)}$$

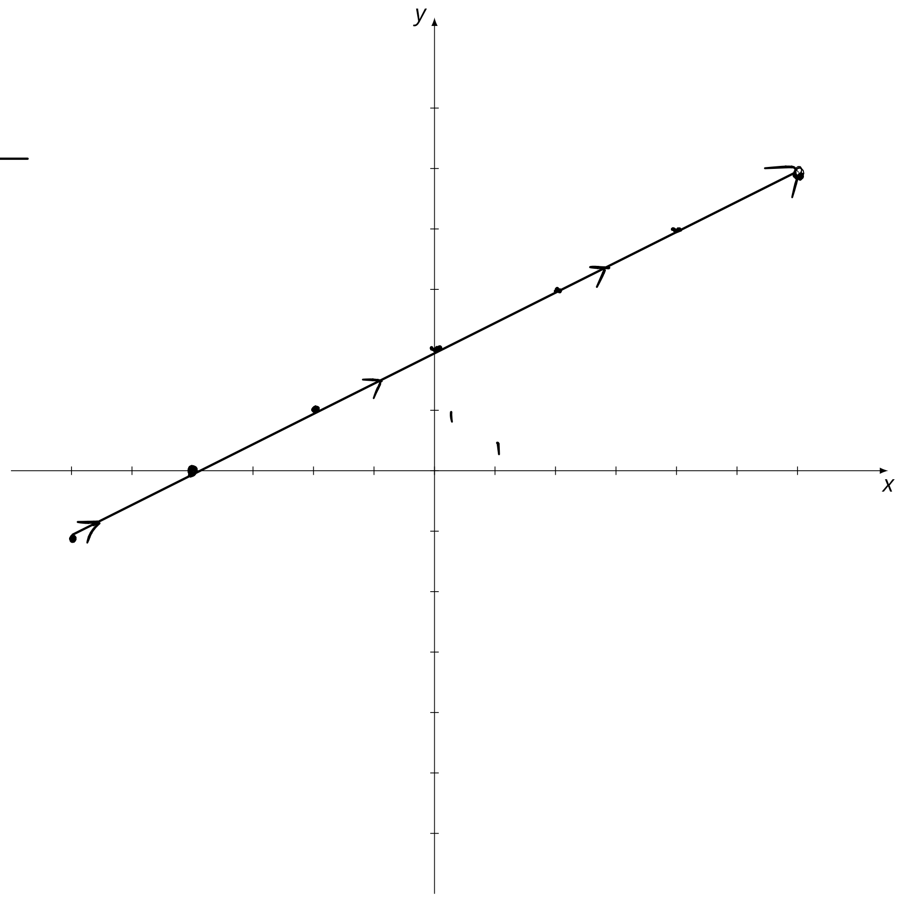
$$= \boxed{\tan(y)}$$

11. Here is a pair of parametric equations

$$x = 2t \qquad y = t + 2$$

(a) Sketch the curve represented by the equations.

t	x	y
-3	-6	-1
-2	-4	0
-1	-2	1
0	0	2
1	2	3
2	4	4
3	6	5



(b) Find a rectangular coordinate equation for the curve by eliminating the parameter.

$y = t + 2$

$\downarrow$

$t = y - 2$

$\downarrow$ substitute

$\boxed{x = 2(y-2)}$

12. Answer the following. Do not leave negative exponents.

(a) Simplify

$$\frac{1}{x-1}+\frac{1}{x+1}-\frac{2}{x}$$

LCD $= \frac{x(x+1)}{x(x+1)}\frac{1}{x-1} + \frac{x(x-1)}{x(x-1)}\frac{1}{x+1} - \frac{(x-1)(x+1)}{(x-1)(x+1)}\frac{2}{x}$

frac law #1, dist law, A^2-B^2 $= \frac{x^2+x}{x(x+1)(x-1)} + \frac{x^2-x}{x(x+1)(x-1)} - \frac{2(x^2-1)}{x(x+1)(x-1)}$

frac law #3 dist $= \frac{x^2+x+x^2-x-2x^2+2}{x(x+1)(x-1)}$

$= \boxed{\frac{2}{x(x+1)(x-1)}}$

(b) Simplify

$$x^2y\left(\frac{x+1}{y}\right)^{-2}\left(\sqrt{\frac{x}{y}}\right)^4$$

$= x^2y\left(\frac{y}{x+1}\right)^2\left(\left(\frac{x}{y}\right)^{\frac{1}{2}}\right)^4$

$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$, $(a^n)^m = a^{nm}$ $= x^2y\frac{y^2}{(x+1)^2}\left(\frac{x}{y}\right)^2$

frac law 1, $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ $= \frac{x^2y\cdot\cancel{y^2}}{(x+1)^2}\cdot\frac{x^2}{\cancel{y^2}}$

frac law 1, 5 $= \frac{x^2x^2y}{(x+1)^2}$

$a^na^m = a^{n+m}$ $= \boxed{\frac{x^4y}{(x+1)^2}}$

(c) If

$$f(x) = x^2 + 1 \qquad g(x) = 2x^3 \qquad h(x) = 2x - 1 \qquad k(x) = 6x^2$$

fully expand and simplify the following expressions:

i. $f(x)g(x) + h(x)k(x)$

$$= (x^2+1)\,2x^3 + (2x-1)\,6x^2$$

$$\overset{\text{dist}}{=} 2x^5 + 2x^3 + 12x^3 - 6x^2$$

$$= \boxed{2x^5 + 14x^3 - 6x^2}$$

ii. $\dfrac{g(x)f(x) - k(x)h(x)}{[k(x)]^2}$ $= \dfrac{2x^3(x^2+1) - 6x^2(2x-1)}{(6x^2)^2}$

$$\overset{\text{dist law}}{=} \frac{2x^5 + 2x^3 - 12x^3 + 6x^2}{6^2 \cdot (x^2)^2}$$

$(a \cdot b)^n = a^n b^n$

$(a^n)^m = a^{nm}$

$$= \boxed{\frac{2x^5 - 10x^3 + 6x^2}{36x^4}}$$